Malaysian Journal of Mathematical Sciences 19(1): 147–161 (2025)
https://doi.org/10.47836/mjms.19.1.08

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

Inventory Model for Defective and Deteriorating Items With Two Production Rates, Stochastic Demand, and Payment Delay Within A Finite Horizon

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Received: 31 May 2024
Accepted: 2 October 2024

Abstract

An inventory production model is developed specifically for managing deteriorating items, encompassing the presence of defective items within the production process. Within this framework, demand is modelled as stochastic, following the Wiener process. Notably, the planning horizon is constrained to a finite timeframe. Additionally, the model incorporates the assumption of shortages, two distinct production rates, reflecting the dynamic nature of real-world manufacturing operations. Furthermore, the impact of delays in payments for raw materials, from the manufacturer to the supplier, is also accounted. The resolution of the stochastic integral is achieved through the rigorous application of the Riemann integral method. The model is solved, and the optimal cost is calculated demonstrating effective management strategies. A numerical example is solved, followed by sensitivity analysis to illustrate practical applications. This article offers an approach to optimize inventory management under complex real-world constraints.

Keywords: stochastic demand; Wiener process; two production rates; defective items; credit financing.

1 Introduction

In traditional inventory models, demand was often considered to be deterministic. However, in real-life situations, demand can be unpredictable, posing challenges for firms seeking to optimize their inventory. Ritchie [25] introduced an Economic Order Quantity (EOQ) model for non-stationary demand, pioneering research in this area. Despite its importance, research into stochastic inventory systems remains limited. Ma et al. [18] conducted a literature review of stochastic models, focusing on determining optimal replenishment order timing and quantities amidst demand uncertainty. Uncertainty in demand not only impacts replenishment decisions but also affects production rates, which can vary Ben-Daya et al. [3]. Yu and Dong [38] also explored a two-stage inventory production system under demand randomness. Consequently, maintaining inventory levels while reducing total costs becomes challenging. Addressing this challenge, Shah et al. [31] proposed an inventory model for dynamic demand.

In companies, efficient inventory management is crucial to minimize total costs. For certain products with high holding costs and deterioration rates but low shortage costs, it is profitable to consider shortages. Consequently, the realm of inventory management, particularly concerning shortages, has been extensively explored in academic literature. The pioneering work by Bose et al. [4] introduced deteriorating items considering shortages in the context of inflation and time discounting. Sometimes, no lost sales occur during shortages. Assuming this, Keshavarzfard et al. [16] considered a model in which shortages are fully backlogged. Various scholars have contributed innovative approaches to address different aspects of this domain. Dye et al. [9] investigated pricing strategies for deteriorating items characterized by completely backlogged shortages. Nobil et al. [24] also proposed an inventory model considering shortages with warm-up process.

Furthermore, in manufacturing, defective items may arise during production, necessitating quality screening before sale. Considering this, Salameh and Jaber [26] developed a model accounting for a fraction of items being of poor quality, a phenomenon dependent on production rates. Manna and Shaikh [20] also formulated an inventory model considering imperfect production processes for deteriorating items. Other authors, such as Salas-Navarro et al. [27], also considered imperfect items with a remanufacturing process. Bai et al. [1] introduced an inventory model for deteriorable & damageable items to determine the optimum ordering quantity and backorder level. Furthermore, Dwicahyani et al. [7] considered a closed-loop system inventory model considering imperfect manufacturing. Nobil et al. [23] also formulated an economic production model considering imperfect manufacturing. Furthermore, Nobil et al. [22] formulated an inventory production model considering both shortages and defective manufacturing.

Multiple production rates allow firms to optimize production workflows and meet customer demand efficiently, especially when certain production processes have predefined schedules or time constraints. Additionally, in the case of a mechanical breakdown or a shortage of raw materials particularly during specific seasons, the production rate could significantly drop. Assuming these scenarios, production rates may change within a firm. Sekar et al. [29] assumed a three-level production inventory model for defective and deteriorating items. Furthermore, Jauhari et al. [13] considered a model with an adjustable production rate to prevent deterioration. To improve production-inventory decisions under stochastic demand, Jauhari [12] considered a model assuming imperfect production. Misni and Lee [21] also presented an inventory production model and solved it using the Modified Harmony Search (MHS) algorithm.

Additionally, buyers may face situations where full payment is not immediately possible, prompting suppliers to offer trade credit in the form of permissible payment delays [11]. Liao et al. [17] proposed a model for deteriorating items, incorporating permissible payment delays. Additionally, understanding the timing of payments between the manufacturer and supplier is important for managing cash flows and maintaining healthy relationships in the supply chain. By specifying payment terms within each production cycle, firms can effectively plan their financial obligations and ensure smooth operations. Assuming this, Mandal et al. [19] derived an integrated inventory model for deteriorating products, considering credit policies and shortages.

Moreover, in real-world scenerios, considering credit financing increases demand and helps in growing business. Motivated by this, Taleizadeh et al. [32] introduced an EPQ model considering credit financing in supply chain management. Authors like Jayanthi [14] also proposed innovative approaches to solve inventory models, assuming the circumstances of permissible delays in payments for deteriorating items. Duan et al. [6] formulated a production model for deteriorating items within a finite planning horizon, assuming demand follows a Wiener Process.

In this model, only a single item is considered to simplify the analysis and provide a clear focus on optimizing inventory and production strategies. Additionally, shortages and defective or deteriorating items are included to ensure the model accurately reflects practical challenges and their impacts on inventory management. Lead time is considered negligible, allowing for an immediate understanding of inventory dynamics without the added complexity of replenishment delays. Furthermore, a finite planning horizon is assumed to provide a defined timeframe for analysis, facilitating actionable insights and decision-making. Demand is treated as stochastic and modelled by the Wiener process to capture the inherent variability and unpredictability in real-world demand, making the model more realistic. Additionally, two production rates are assumed to account for varying production capacities or strategies, and it is also assumed that payments are made at the end of each cycle to reflect typical financial practices, impacting cash flow management.

2 Related Works

Many authors have provided economic inventory models. For example, Sebatjane and Adetunji [28] presented an economic ordering quantity model, whereas You and Grossmann [37] proposed a model for supply chain design with stochastic inventory management. Some of the research articles are discussed in Table 1.

Table 1 provides a comparative overview of various studies on inventory management, each addressing different aspects such as stochastic demand, deterioration of inventory, defective items, multiple production rates, permissible payment delays, and the planning horizon. For instance, Xia et al. [36] focus solely on multiple production rates, while Teng et al. [34] consider deterioration and permissible payment delays, but exclude stochastic demand and multiple production rates. Shah et al. [30] explored permissible payment delays and defective items but do not include stochastic demand or multiple production rates. On the other hand, Jose and Nair [15] address stochastic demand and multiple production rates but lack permissible payment delays and a fixed planning period. Each study thus contributes to the field of inventory management by examining specific combinations of these factors.

Despite these contributions, gaps remain in integrating stochastic demand, production variability, quality issues, and payment delays within a finite inventory framework. Integrating these scenarios reflects practical challenges faced by manufacturers in managing quality control, cash flow through trade credit, and production variability effectively.

Table 1: A brief literature review of inventory models in a finite planning horizon.

Article	**Stochastic Demand**	**Deterioration**	**Defective Items**	**Multiple Production Rates**	**Permissible Payments Delay**	**Finite Planning Horizon**	**Comparison with Current Study**
Xia et al. [36]	✗	✗	✗	✓	✗	✗	Only considers multiple production rates.
Teng et al. [34]	✗	✓	✗	✗	✓	✗	Addresses deterioration and permissible payments delay but lacks stochastic demand and multiple production rates.
Dye [8]	✗	✓	✗	✗	✓	✓	Considers permissible payment delays for deteriorating items but does not include stochastic demand or multiple production rates, despite the possibility of some items being manufactured defective.
Shah et al. [30]	✗	✓	✗	✗	✓	✓	
Taleizadeh and Nematollahi [33]	✗	✓	✗	✗	✓	✓	
Wu and Zhao [35]	✗	✓	✗	✗	✓	✓	
Jose and Nair [15]	✓	✓	✗	✓	✗	✗	Considers stochastic demand and multiple production rates but lacks permissible payments delay, and a fixed planning period.
Duan et al. [6]	✓	✓	✗	✗	✗	✓	Considers stochastic demand under a finite planning horizon but lacks multiple production rates, and credit financing.
Das et al. [5]	✗	✓	✓	✗	✓	✗	Addresses credit financing but lacks stochastic demand, and multiple production rates under a finite horizon.
Barman and Mahata [2]	✓	✗	✓	✗	✗	✓	Considers stochastic demand but lacks multiple production rates, and permissible payments delay.
Garg et al. [10]	✗	✓	✓	✗	✗	✗	Considers defective and deteriorating items only under an infinite horizon and does not account for stochastic demand.

From the review of the existing literature review, it is evident that no studies are done for an inventory model for deteriorating and defective items having stochastic demand and two production rates in a finite planning horizon with permissible delay in payments. This study aims to address these gaps by developing a comprehensive inventory model that considers stochastic demand following a Wiener process, variable production rates, quality screening, shortages, and payment delays. By filling these gaps, the study seeks to provide practical insights for optimizing inventory management under complex real-world constraints.

This study mainly aims to answer the following questions:

1. How to model stochastic demand using Weiner process?
2. How to find optimal replenishment decision under multiple production rates.
3. Effect of trade credit in inventory management.

3 Methodology

3.1 Assumptions

- Only a single item is considered.
- Shortages are considered.
- Lead time is considered negligible.
- Planning horizon is finite.
- Demand is stochastic in nature and follows the Wiener process.
- Defective and deteriorating items are considered.
- Two production rates are considered.
- The manufacturer pays the supplier at the end of each cycle.
- Production with rate P_2 starts after a fixed time and continues until a fixed time in each cycle.

3.2 Notations

C	:	Ordering cost per cycle (in $\$$).
C_1	:	Unit purchasing cost of raw material (in $\$$).
K	:	Screening cost of manufactured goods per cycle (in $\$$).
R	:	Per unit remanufacturing cost of defective items (in $\$$).
$D(t)$	:	Demand rate at time t in units.
μ	:	The average market potential in units.
σ	:	The instability in demand in units.
$W(t)$	:	The standard Wiener process at time t (Brownian Motion) in units.
β	:	Fraction of production of defective items in units.
H	:	The planning horizon in days.
TC	:	Total cost of manufacturer (in $\$$).
h	:	Holding cost/unit/day (in $\$$).
s	:	Shortages cost/unit (in $\$$).
Q_{i+1}	:	The total manufactured quantity in units in the $(i+1)^{\text{th}}$ cycle at time t_i.
k	:	Total number of replenishment cycles.
α	:	Deterioration rate per unit time $(0 < \alpha < 1)$.
$I_{i+1}(t)$	:	Inventory level in units in the $(i+1)^{\text{th}}$ cycle at time t.
$S_{i+1}(t)$	:	Shortage level in units in the $(i+1)^{\text{th}}$ cycle at time$t$.
P_1, P_2	:	Production rates, i.e., number of produced units per unit time.
t_i	:	i^{th} replenishment time, i.e., the time at which production starts with rate P_1 in the $(i+1)^{\text{th}}$ cycle.
t_{p1i}	:	The time at which production starts with rate P_2 in the $(i+1)^{\text{th}}$ cycle.
t_{p2i}	:	Time at which production stops in the $(i+1)^{\text{th}}$ cycle.
t_{si}	:	The time at which shortages start in the $(i+1)^{\text{th}}$ cycle.
i_c	:	Interest rate per unit purchasing cost of raw material per unit time.
M	:	Permissible delay time in payments in days.

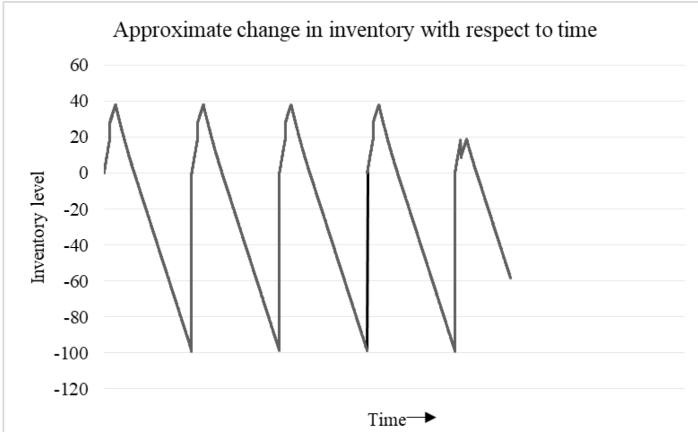

Figure 1: Pictorial representation of change in inventory with respect to time in a cycle.

4 Mathematical Model

Since the demand rate is stochastic thus stochastic change in potential demand can be written as the following differential equation,

$$\frac{dD(t)}{dt} = \mu - \sigma \frac{dW(t)}{dt}. \tag{1}$$

Since the Wiener process is zero at the initial point, in the $(i+1)^{\text{th}}$ cycle, consider production starting at the beginning of the cycle with a production rate of P_1 until time t_{p1i}. After that, the production rate changes to P_2 in response to demand variability. Subsequently, production ceases at time t_{p2i}, and the change in inventory level occurs solely due to demand and the deterioration rate. At time t_{si}, inventory reaches zero, and shortages begin. Thus, the equation for the change in inventory level at any time t can be written as follows,

$$\frac{dI_{i+1}(t)}{dt} = -\alpha I_{i+1}(t) + P_1(1-\beta) - \mu + \sigma \frac{dW(t)}{dt}, \qquad t_i \le t \le t_{p1i}, \tag{2}$$

$$\frac{dI_{i+1}(t)}{dt} = -\alpha I_{i+1}(t) + P_2(1-\beta) - \mu + \sigma \frac{dW(t)}{dt}, \qquad t_{p1i} \le t \le t_{p2i}, \tag{3}$$

$$\frac{dI_{i+1}(t)}{dt} = -\alpha I_{i+1}(t) - \mu + \sigma \frac{dW(t)}{dt}, \qquad t_{p2i} \le t \le t_{si}, \tag{4}$$

$$\frac{dS_{i+1}(t)}{dt} = \mu - \sigma \frac{dW(t)}{dt}, \qquad t_{si} \le t \le t_{i+1}. \tag{5}$$

Boundary conditions,

$$I_{i+1}(t_i) = 0, \qquad I_{i+1}(t_{si}) = 0, \qquad S_{i+1}(t_{si}) = 0, \qquad W(t_i) = 0.$$

Solving above equations, we get

$$I_{i+1}(t) = \left(1 - e^{\alpha(t_i - t)}\right)\left[\frac{P_1(1-\beta) - \mu}{\alpha}\right] + \sigma e^{-\alpha t} \int_{t_i}^{t} e^{\alpha t}\, dW(t), \qquad t_i \le t \le t_{p1i}, \tag{6}$$

$$I_{i+1}(t) = \left(e^{\alpha(t_{si}-t_{p2i})}-1\right)\frac{\mu}{\alpha} - \sigma e^{-\alpha t_{p2i}}\int_{t_{p2i}}^{t_{si}} e^{\alpha t}\,dW(t) + \left(1-e^{\alpha(t_{p2i}-t)}\right)\left[\frac{P_2(1-\beta)-\mu}{\alpha}\right] - \sigma e^{-\alpha t}\int_t^{t_{p2i}} e^{\alpha t}\,dW(t), \qquad t_{p1i}\le t\le t_{p2i}, \tag{7}$$

$$I_{i+1}(t) = \left(e^{\alpha(t_{si}-t)}-1\right)\frac{\mu}{\alpha} - \sigma e^{-\alpha t}\int_t^{t_{si}} e^{\alpha t}\,dW(t), \qquad t_{p2i}\le t\le t_{si}, \tag{8}$$

$$S_{i+1}(t) = \mu(t-t_{si}) - \sigma W(t), \qquad t_{si}\le t\le t_{i+1}. \tag{9}$$

Now, the total produced inventory, including the defective items, in the time interval $[t_i, t_{p2i}]$ is equal to

$$P_1(t_{p1i}-t_i) + P_2(t_{p2i}-t_{p1i}).$$

Thus,

$TC =$ holding cost + Purchasing cost of raw material + Screening cost + Remanufacturing cost of defective items + fixed ordering cost + Interest paid by the manufacturer + Shortages cost.

Which implies,

$$TC = kC + K + \text{Interest paid by the manufacturer} + \sum_{i=0}^{k-1}\left(h\int_{t_i}^{t_{si}} I_{i+1}(t)\,dt + s\int_{t_{si}}^{t_{i+1}} S_{i+1}(t)\,dt + (C_1+R\beta)\Big(P_1(t_{p1i}-t_i)+P_2(t_{p2i}-t_{p1i})\Big)\right). \tag{10}$$

Let,

$$TC^* = kC + K + \sum_{i=0}^{k-1}\left(h\int_{t_i}^{t_{si}} I_{i+1}(t)\,dt + s\int_{t_{si}}^{t_{i+1}} S_{i+1}(t)\,dt + (C_1+R\beta)\Big(P_1(t_{p1i}-t_i)+P_2(t_{p2i}-t_{p1i})\Big)\right). \tag{11}$$

Now for some fixed k, to determine the optimal ordering interval, we must have

$$\frac{\partial TC^*}{\partial t_i} = 0, \qquad \frac{\partial TC^*}{\partial t_{si}} = 0, \qquad \text{for} \quad i = 0,\dots,k-1, \quad \text{with} \quad t_0 = 0, \quad t_k = H.$$

Now, differentiating TC^* and by applying the Leibniz Rule of differentiation, we get

$$\frac{\partial TC^*}{\partial t_i} = -(C_1+R\beta)P_1 + h\frac{(P_1(1-\beta)-\mu+\sigma)}{\alpha}\left(-1+e^{\alpha(t_i-t_{p1i})}\right) + S\mu(t_i - t_{s(i-1)}) = 0. \tag{12}$$

for each $i = 1,\dots,k-1$, with $t_0 = 0$, $t_k = H$, and

$$\frac{\partial TC^*}{\partial t_{si}} = h\left((\mu-\sigma)e^{\alpha(t_{si}-t_{p2i})}(t_{p2i}-t_{p1i}) + \frac{\mu-\sigma}{\alpha}\left(e^{\alpha(t_{si}-t_{p2i})}-1\right)\right) + S\mu(t_{si}-t_{i+1}) = 0, \tag{13}$$

for each $i = 1,\dots,k-1$, with $t_0 = 0$, $t_k = H$.

Now, by applying assumption (9) in (12), we get

$$\frac{\partial TC^*}{\partial t_i} = \text{some constant quantity} + (\text{some constant quantity})t_i + (\text{some constant quantity})t_{s(i-1)} = 0, \tag{14}$$

for each $i = 1, \dots, k-1$, with $t_0 = 0$, $t_k = H$. From (14), corresponding to unique $t_{s(i-1)}$, we will get a unique t_i.

Moreover, by applying assumption (9) in (13), we get

$$\frac{\partial TC^*}{\partial t_{si}} = (\text{some constant quantity})e^{\alpha(t_{si}-t_i)} + (\text{some constant quantity})t_{si} + (\text{some constant quantity})t_{i+1} = 0, \tag{15}$$

for each $i = 1, \dots, k-1$, with $t_0 = 0$, $t_k = H$.

Using (14) in (15), we have

$$(\text{some constant quantity})e^{\alpha(t_{si}-t_i)} + (\text{some constant quantity})t_{si} - \text{some constant quantity} = 0. \tag{16}$$

Now, for $i = 0$, $t_0 = 0$, using this in (16), we get

$$(\text{some constant quantity})e^{\alpha t_{s0}} = \text{some constant quantity} - \text{some constant quantity}\, t_{s0}. \tag{17}$$

Since, left side of (17) is an exponential function and right side is a line in variable t_{s0}. The line and exponential function can intersect at at-most one point on the real axis. Thus, (17) can have at most one solution. Hence, there exists at most one t_{s0} such that $t_0 < t_{s0} \leq H$.

Thus, two cases arise;

Case 1: When (17) has no solution:
In this case, using (14), we can see that t_1 does not exist.

Case 2: When (17) has one solution:
In this case, using (14), we get unique t_1.

Putting this value of t_1 in (15), and by proceeding as above, we get at most one t_{s1}. By proceeding further, we can get at most one solution of the simultaneous (14) and (15) such that,

$$0 = t_0 \leq t_1 \leq \cdots \leq t_{i-1} \leq t_i \leq t_{i+1} = H.$$

Furthermore,

$$\frac{\partial^2 TC^*}{\partial t_i^2} = \Big(P_1(1-\beta) - \mu + \sigma\Big)e^{\alpha(t_i - t_{p1i})} + S\mu > 0, \tag{18}$$

$$\frac{\partial^2 TC^*}{\partial t_i\, \partial t_{i-1}} = 0 = \frac{\partial^2 TC^*}{\partial t_i\, \partial t_{i+1}}, \tag{19}$$

and

$$\frac{\partial^2 TC^*}{\partial t_{si}^2} = h\left(\alpha(\mu-\sigma)e^{\alpha(t_{si}-t_{p2i})}(t_{p2i} - t_{p1i}) + (\mu-\sigma)e^{\alpha(t_{si}-t_{p2i})}\right) + S\mu t_{si} > 0, \tag{20}$$

$$\frac{\partial^2 TC^*}{\partial t_{si}\, \partial t_{s(i-1)}} = 0 = \frac{\partial^2 TC^*}{\partial t_{si}\, \partial t_{s(i+1)}}. \tag{21}$$

Since, both the hessian matrices are positive diagonal matrices and thus positive definite. Thus, the solution if exists will give global minima of TC^*. Now,

Case A: When $M \leq t_{i+1}$,

$$\text{Interest paid by the manufacturer} = C_1 i_c \Big(P_1(t_{p1i} - t_i) + P_2(t_{p2i} - t_{p1i}) \Big)(t_{i+1} - M_{i+1}). \tag{22}$$

Case B: When‘$M > t_{i+1}$,

$$\text{Interest paid by the manufacturer} = 0. \tag{23}$$

Theorem 4.1. *Equation* (10) *is convex in* k *with* $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_k = H$.

Proof. Since interest paid by the manufacturer is linear function so it is both concave and convex. And since, quadratic function is also convex provided the coefficient of leading term is positive. Also, since, sum of two convex functions is convex. Thus, for both the cases A and B, it is sufficient to prove convexity if,

$$\sum_{i=0}^{k-1} \int_{t_i}^{t_{si}} I_{i+1}(t)\, dt \text{ is convex.}$$

For fixed k, the interval $(0, H)$ is splitted into k sections. Now, suppose $k = k + 1$, the interval $(0, H)$ is splitted into $k + 1$ sections i.e. the interval is arbitrary fitted at one point in accordance with the previous point.

Now let,

$$f(k, 0, H) = \sum_{i=0}^{k-1} \int_{t_i}^{t_{si}} I_{i+1}(t)\, dt. \tag{24}$$

Then, $f(k, 0, H)$ is convex in k if,

$$2f(k, 0, H) \leq f(k - 1, 0, H) + f(k + 1, 0, H). \tag{25}$$

Simplifying (25) by putting the stochastic integral as a Riemann summation and taking,

$$dW(t) = \mathbb{E}\,(\delta t)^{1/2},$$

where $\mathbb{E}$ is a normally distributed random variable with mean 0 and variance 1, and δt is a small step size.

Since e^t is a convex function, we get

$$f(k, 0, H) - f(k - 1, 0, H) - \Big[f(k + 1, 0, H) - f(k, 0, H) \Big] \leq 0. \tag{26}$$

□

4.1 Algorithm

- Approximate the stochastic integral using the Riemann sum by dividing the integration interval into very small partitions.
- Generate normally distributed random variables with a mean of 0 and variance of 1 to introduce stochasticity into the Wiener process over each time interval.

- Change the Wiener process in the form of time with the help of the generated random variables, and update the Wiener process $W(t_i)$ at each time step t_i using

$$W(t_i) = W(t_{i-1}) + \mathbb{E}\,(t_i - t_{i-1})^{1/2}\,,$$

  where $W(t_{i-1})$ is the Wiener process at time step t_{i-1}.

- Find the optimum time interval for fixed n and proceed.

4.2 Different cases for different conditions

- When the production rate is unique, consider $P_1 = P_2$.
- When there is no shortage, consider $t_{si} = t_{i+1}$.

4.3 Numerical example

Example 4.1. *$\sigma = 0.5$, $\mu = 10$ units per day, $\alpha = 0.1$, $H = 70$ days, $R = 0.50$ \$/unit, $\beta = 0.01$, $h = 1$ \$/unit/day, $S = 0.5$ \$/unit/day, $M = 1$ day, $P_1 = 30$ units/day and $P_2 = 20$ units/day, $C_1 = 1$ \$/unit, $K = 2$ \$/cycle, $C = 2$ \$/cycle, and $i_c = 0.1$ per unit time. The production with rate P_2 starts after one day of each cycle and ends one day after starting. Find the optimum number of cycles.*

Answer: From Table 2, it is clear that the optimum number of cycles is 5. Additionally, Table 2 mentions the optimum inventory replenishment times for each k. Furthermore, Figure 2 depicts the approximate change in inventory level over time for $k = 5$. Figure 3 illustrates that the total cost decreases convexly with the number of cycles.

The total cost for $k = 5$ is equals to the total deterministic cost + the approximate stochastic change in total cost $= 2855.08 - 10.2906 = 2844.79$.

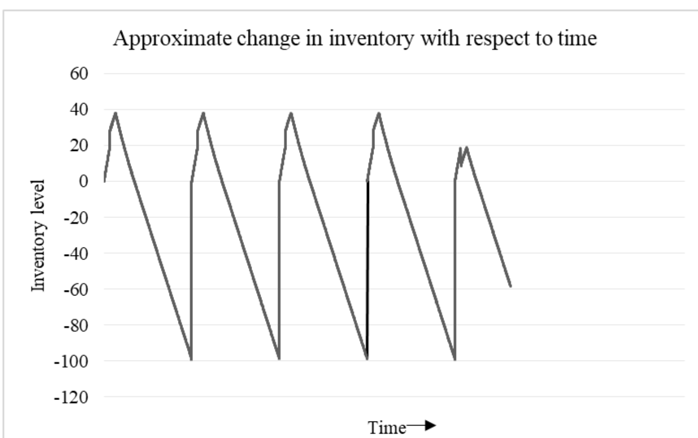

Figure 2: Approximate change in inventory level with respect to time.

Table 2: Optimum time interval and optimum total cost with respect to k (Note that t_i and t_{si} are in days).

k	t_{s0}	t_1	t_{s1}	t_2	t_{s2}	t_3	t_{s3}	t_4	t_{s4}	t_5	TC^*
1	14.686	70	-	-	-	-	-	-	-	-	9595.47
2	5.232	15.107	27.844	70	-	-	-	-	-	-	6360.38
3	5.232	15.107	20.339	30.213	40.609	70	-	-	-	-	4180.05
4	5.232	15.107	20.339	30.213	35.446	45.32	52.81	70	-	-	3362.32
5	5.232	15.107	20.339	30.213	35.446	45.32	50.552	60.427	64.156	70	2844.79
6		Solution does not exist				-	-	-	-	-	-

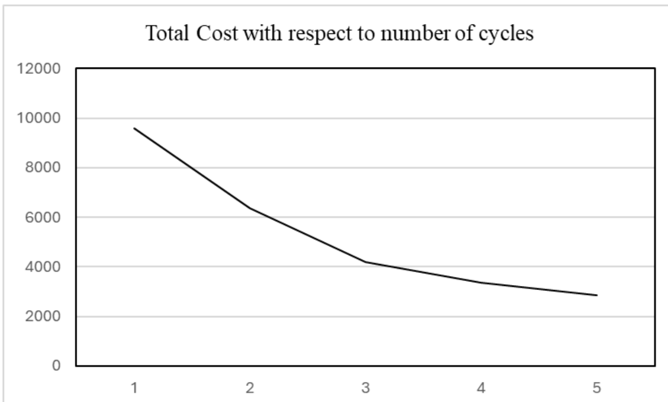

Figure 3: From this figure, it is observed that total cost decreases with increase in number of replenishment cycles.

4.4 Sensitivity analysis with respect to some variables

Table 3: Sensitivity analysis with respect to initial production rate, deterioration rate, holding cost, shortages cost, and the production time.

P_1	α	S	h	t_{p1}	**Deterministic Total Cost**
29	0.10	0.5	1.00	$t_i + 1.0$	2778.61
31	0.10	0.5	1.00	$t_i + 1.0$	2974.87
30	0.09	0.5	1.00	$t_i + 1.0$	2787.64
30	0.11	0.5	1.00	$t_i + 1.0$	2983.69
30	0.10	0.5	1.00	$t_i + 1.1$	2943.23
30	0.10	0.5	1.00	$t_i + 0.9$	2815.11
30	0.10	0.4	1.00	$t_i + 1.0$	No Optima
30	0.10	0.6	1.00	$t_i + 1.0$	3145.25
30	0.10	0.5	0.75	$t_i + 1.0$	2708.45
30	0.10	0.5	1.25	$t_i + 1.0$	3038.29

From Table 3, it is evident that the total cost is directly proportional to the production rate P_1, deterioration rate α, shortage cost S, holding cost h, and production time t_{p1}. Since the variable P_2 also describes the production rate, it is sufficient to conduct sensitivity analysis on P_1 alone rather than both P_1 and P_2. Furthermore, the total cost equation itself shows that the total cost is directly proportional to C_1, K, and C. Additionally, as we have seen from the sensitivity of S, it is evident that changing the values of variables may also change the optimum number of cycles. Furthermore, obtaining an optimum for a particular value of a variable does not guarantee that we will get an optimum for other values of that variable. Figure 2 shows the approximate change in inventory level with time.

5 Managerial Insights

The article on inventory management for deteriorating items with stochastic demand, shortages, and varying production rates provides several critical managerial insights. Firstly, it emphasizes the importance of optimizing ordering cycles under conditions where shortages may occur. Managers are encouraged to find the balance between minimizing costs and ensuring sufficient inventory levels to meet demand fluctuations effectively. Secondly, by optimizing production schedules and adopting technologies to minimize deterioration, businesses can reduce holding costs while maintaining product quality. Thirdly, the adoption of stochastic demand modelling allows managers to better forecast demand variability. Managing payment delays for raw materials is critical; strategies like negotiating payment terms or diversifying suppliers help maintain smooth operations and cash flow. Furthermore, the application of sensitivity analysis enables managers to assess how changes in production rates, deterioration rates, and other factors impact total costs. This analytical approach guides decision-making by identifying critical variables and optimizing operational strategies accordingly.

6 Conclusion

In this article, it is demonstrated that under assumed conditions, the total cost exhibits convex behaviour with respect to the number of cycles. Furthermore, sensitivity analysis reveals that the total cost is directly proportional to the production rate, the deterioration rate, and the production time. Moreover, it is evident that the total cost is highly sensitive to changes in the deterioration rate. It is also established that in the presence of shortages, an optimal ordering cycle length may or may not exist. However, if such an optimum exists, the local optima of these ordering intervals are also global optima. Effective implementation of this model requires accurate historical data on demand patterns, production rates, and deterioration rates. Obtaining and maintaining such data can be challenging, especially in dynamic environments. A limitation of this work lies in the consideration of shortages, where the existence of an optimum ordering cycle length is not guaranteed. Additionally, the existence and value of optima are sensitive to the values of variables, therefore, one should be careful when choosing particular values for these variables. Future research could explore approximating stochastic demand models to ensure the existence of an optimal cycle length under such conditions. Furthermore, incorporating real-time data could enhance the model's robustness and applicability in various scenarios.

Acknowledgement The authors sincerely acknowledge the support and contributions of all individuals and institutions who assisted in the completion of this work.

Conflicts of Interest We wish to confirm that there is no conflict of interest associated with this article that could have influenced the outcome.

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